CHOICE OF SPACE AND TIME STEPS IN CALCULATING TEMPERATURE DISTRIBUTIONS

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As an example of the use of the explicit finite-difference scheme for calculating the temperature distribution in an infinite plate we discuss a method of constructing networks which leads to stable solutions for a specified computational accuracy.

It is well known that the nonstationary propagation of heat in a wall can be described by a system of explicit finite-difference equations. The stability and convergence of the solution of such a system are determined by the magnitudes and the ratio of the space and time steps used in the numerical integration of the system [1-4].

For the simplest three-point scheme the magnitudes of the temperatures ϑ_i and $\vartheta_i(\Delta\text{Fo})$ corresponding to the times Fo and Fo + ΔFo are connected by the following relation, written in dimensionless form

$$\vartheta_i(\Delta\text{Fo}) = \vartheta_{i-1}A_i\Delta\text{Fo} + \vartheta_i[1 - \Delta\text{Fo}(A_i + B_i)] + \vartheta_{i+1}B_i\Delta\text{Fo}. \tag{1}$$

It is assumed that the thermal flux is uniform and that the ambient temperature and the thermophysical characteristics of the wall are constant. The coefficients A_i, B_i are found by applying the heat balance condition to the elementary parts into which the wall is decomposed for the calculation. The number of these parts m determines the size of the space steps.

The requirement that the solution of a system of equations of type (1) be stable for all nodal points is satisfied if

$$\Delta\text{Fo} \leqslant \frac{1}{A_i + B_i}, \quad 0 \leqslant i \leqslant m. \tag{2}$$

We note that points with number i = 0 and i = m belong to the boundary surfaces of the wall.

Condition (2) establishes the largest admissible value of a time step ΔFo. Another natural limitation is imposed by the required accuracy of the calculation of the temperature distribution. High accuracy in nonstationary thermal problems can be achieved only for sufficiently large values of m. At the same time increasing m leads to a sharp increase in computing time. If for m = 10 a computer of the M-20 type [5] requires one second to compute the temperature distribution in the wall to a value of the Fourier number Fo = 1, then for m = 80-100 a similar calculation requires 10-20 minutes of machine time. It should be noted that the design of a heat engine generally involves the study of dozens of structural variations. Therefore, a reasonable choice of time and space steps in calculating temperature distributions is of considerable value.

Analysis of the effect of m on the accuracy of the calculation of the temperature is most conveniently performed for an infinite plate with boundary conditions of the first kind. In this case Eq. (2) takes the form

$$\Delta\text{Fo} \leqslant \frac{1}{2m^2}. \tag{3}$$

Equation (3) contains the single parameter m.

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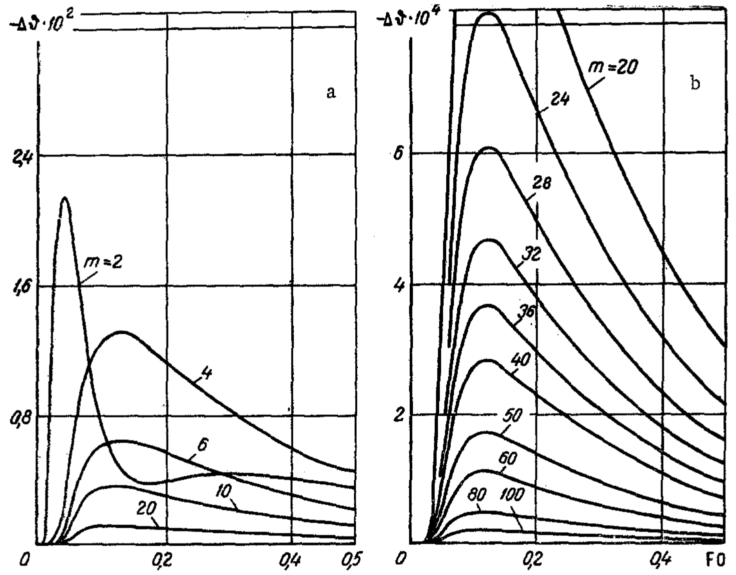

Fig. 1. Dependence of the error $\Delta\vartheta = \vartheta - \vartheta_{exact}$ on Fo a) for small and b) for large m.

The initial temperature of the wall is taken as zero at all points. The boundary conditions can be written in the form

$$\vartheta_0(\mathrm{Fo}) = \vartheta_m(\mathrm{Fo}) = 1.$$

The time step ΔFo in the calculations is determined by the equality (3)

$$\Delta\mathrm{Fo} = \frac{1}{2m^2}\ . \tag{4}$$

For convenience in plotting graphs the values obtained for ΔFo are reduced to the nearest multiple of 0.01, the value chosen for the time step in the printout.

The accuracy of the calculation can be checked, for example, by the value of the temperature at the center of the plate ϑ(Fo). We denote the exact value of the check temperature by ϑ_{exact}.

Figure 1 shows the error $\Delta\vartheta = \vartheta - \vartheta_{exact}$ as a function of Fo for various values of m. The figure shows that the maximum of the absolute values of the error for $m \geq 4$ occurs for Fo = 0.12-0.14. If a value is set for the maximum admissible error $|\Delta\vartheta|_{max}$, then for each value of Fo it is possible to find the smallest value of m ensuring the required accuracy of the calculation. The resulting graph for the choice of m is shown in Fig. 2.

The curves of m (Fo, $|\Delta\vartheta|_{max}$) give the values of m permitting calculations to a specified accuracy with a minimum expenditure of machine time. The size of the step ΔFo for a chosen m can be determined from Eq. (4). In the range $0 \leq \mathrm{Fo} \leq 1$ the choice of m and ΔFo by the method described halves the calculation time in comparison with the time necessary with a fixed m. Thus if the temperature of a plate is to be determined to three significant figures ($|\Delta\vartheta|_{max} \leq 0.0005$) m must be taken equal to 32 in calculations with a constant step, whereas in calculating with the curve for $|\Delta\vartheta|_{max} \leq 0.0005$ the value of m varies from 32 to 5.

A further decrease in machine time can obviously be achieved if instead of determining the step ΔFo from (4) a relation is used which ensures a smaller error at each step of the calculation, e.g., the relation

$$\Delta\mathrm{Fo} = \frac{1}{6m^2}\ , \tag{5}$$

obtained from

$$l = \frac{h^2}{6a}\ . \tag{6}$$

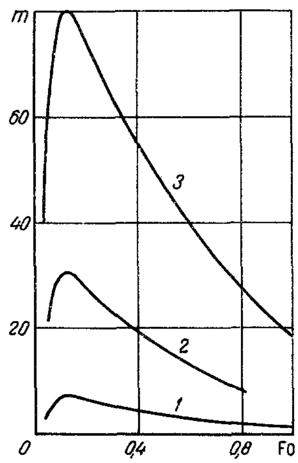

Fig. 2. Graph for determining the number m for various values of the maximum admissible error of the calculation: 1) $|\Delta\vartheta|_{max} \leq 0.5 \cdot 10^{-2}$; 2) $|\Delta\vartheta|_{max} \leq 0.5 \cdot 10^{-3}$; 3) $|\Delta\vartheta|_{max} \leq 0.5 \cdot 10^{-4}$.

As shown in [2, 3] the error of Eq. (6) for a given h is half as large as for the relation

$$l = \frac{h^2}{2a},$$

which corresponds to Eq. (4).

The effectiveness of the method described for choosing the mesh size is verified in an analysis of the heating of solid and hollow cylinders with boundary conditions of the first kind and a plate with boundary conditions of the third kind.

This kind of analysis may turn out to be useful in constructing solutions of similar problems requiring extensive calculations.

NOTATION

Fo	is the dimensionless time (Fourier number);
ΔFo, l	is the length of the time step;
ΔFo_{pr}	is the length of the time step in printout of the machine calculations;
ϑ_i and $\vartheta_i(\Delta Fo)$	are the relative temperatures of the wall at times Fo and Fo + ΔFo, respectively;
$\vartheta_i = [T_i(Fo) - T_0]/(T_a - T_0)$;	
T_0 and $T_i(Fo)$	are the initial and running temperatures of the wall;
T_a	is the ambient temperature;
A_i and B_i	are the constants determined by applying heat balance conditions to elementary parts into which the wall is divided for calculation;
m	is the number of elementary parts;
i	is the ordinal number of nodal point in calculational network, $0 \leq i \leq m$;
$\Delta\vartheta = \vartheta - \vartheta_{exact}$	is the error in the calculating temperature;
ϑ and ϑ_{exact}	are the calculated and exact values of relative temperature at the center of the plate;
$\|\Delta\vartheta\|_{max}$	is the maximum absolute value of admissible error;
h	is the length of the space step;
a	is the thermal diffusivity of material.

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